NEARBY SLOPES AND BOUNDEDNESS FOR ℓ-ADIC SHEAVES IN POSITIVE CHARACTERISTIC

by

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Introduction

Let S be a strictly henselian trait of equal characteristic $p > 0$. As usual, s denotes the closed point of S, k its residue field, $\eta = \operatorname{Spec} K$ the generic point of S, $\overline{K}$ an algebraic closure of K and $\overline{\eta} = \operatorname{Spec} \overline{K}$. Let $f : X \longrightarrow S$ be a morphism of finite type, $\ell \neq p$ a prime number, $\mathcal{F}$ an object of the derived category $D_c^b(X_\eta, \overline{\mathbb{Q}}_\ell)$ of ℓ-adic complexes with bounded and constructible cohomology.

Let $\psi_f^t : D_c^b(X_\eta, \overline{\mathbb{Q}}_\ell) \longrightarrow D_c^b(X_s, \overline{\mathbb{Q}}_\ell)$ be the moderate nearby cycle functor. We say that $r \in \mathbb{R}_{\geqslant 0}$ is a *nearby slope of $\mathcal{F}$ associated to f* if one can find $N \in \operatorname{Sh}_c(\eta, \overline{\mathbb{Q}}_\ell)$ with slope r such that $\psi_f^t(\mathcal{F} \otimes f^* N) \neq 0$. We denote by $\operatorname{Sl}_f^{\mathrm{nb}}(\mathcal{F})$ the set of nearby slopes of $\mathcal{F}$ associated to f.

The main result of [**Tey15**] is a boundedness theorem for the set of nearby slopes of a complex holonomic $\mathcal{D}$-module. The goal of the present (mostly programmatic) paper is to give some motivation for an analogue of this theorem for ℓ-adic sheaves in positive characteristic.

For complex holonomic $\mathcal{D}$-modules, regularity is preserved by push-forward. On the other hand, for a morphism $C' \longrightarrow C$ between smooth curves over k, a tame constructible sheaf on C' may acquire wild ramification by push-forward. If $0 \in C$ is a closed point, the failure of $C' \longrightarrow C$ to preserve tameness above 0 is accounted for by means of the ramification filtration on the absolute Galois group of the function field of the strict henselianization C_0^{sh} of C at 0. Moreover, the Swan conductor at 0 measures to which extent an ℓ-adic constructible sheaf on C fails to be tame at 0.

In higher dimension, both these measures of wild ramification (for a morphism and for a sheaf) are missing in a form that would give a precise meaning to the following question raised in [**Tey14**]

Question 1. — *Let $g : V_1 \longrightarrow V_2$ be a morphism between schemes of finite type over k, and $\mathcal{G} \in D_c^b(V_1, \overline{\mathbb{Q}}_\ell)$. Can one bound the wild ramification of $Rg_*\mathcal{G}$ in terms of the wild ramification of $\mathcal{G}$ and the wild ramification of $g_{|\operatorname{Supp}\mathcal{G}}$?*

Note that in an earlier formulation, "wild ramification of $g_{|\operatorname{Supp}\mathcal{G}}$" was replaced by "wild ramification of g", which cannot hold due to the following example that we owe to Alexander Beilinson: take $f : \mathbb{A}^1_S \longrightarrow S$, $P \in S[t]$ and $i_P : \{P = 0\} \hookrightarrow \mathbb{A}^1_S$. Then $i_{P*}\overline{\mathbb{Q}}_\ell$ is tame but $f_*(i_{P*}\overline{\mathbb{Q}}_\ell)$ has arbitrary big wild ramification as P runs through the set of Eisenstein polynomials.

If $f : X \longrightarrow S$ is proper, proposition 2.2.1 shows that $\mathrm{Sl}^{\mathrm{nb}}_f(\mathcal{F})$ controls the slopes of $H^i(X_{\overline{\eta}}, \mathcal{F})$ for every $i \geqslant 0$. It is thus tempting to take for "wild ramification of $\mathcal{G}$" the nearby slopes of $\mathcal{G}$.

So Question 1 leads to the question of bounding nearby slopes of constructible ℓ-adic sheaves. Note that this question was raised imprudently in [**Tey15**]. It has a negative answer as stated in *loc. it.* since already the constant sheaf $\overline{\mathbb{Q}}_\ell$ has arbitrary big nearby slopes. This is actually good news since for curves, these nearby slopes keep track of the aforementioned ramification filtration [(1)]. Hence, one can use them in higher dimension to quantify the wild ramification of a morphism and in Question 1 take for "wild ramification of $g_{|\operatorname{Supp}\mathcal{G}}$" the nearby slopes of $\overline{\mathbb{Q}}_\ell$ on $\operatorname{Supp}\mathcal{G}$ associated with $g_{|\operatorname{Supp}\mathcal{G}}$ (at least when V_2 is a curve).

To get a good boundedness statement, one has to correct the nearby slopes associated with a morphism by taking into account the maximal nearby slope of $\overline{\mathbb{Q}}_\ell$ associated with the same morphism. That such a maximal slope exists in general is a consequence of the following

Theorem 1. — *Let $f : X \longrightarrow S$ be a morphism of finite type and $\mathcal{F} \in D^b_c(X_\eta, \overline{\mathbb{Q}}_\ell)$. The set $\mathrm{Sl}^{\mathrm{nb}}_f(\mathcal{F})$ is finite.*

The proof of this theorem follows an argument due to Deligne [**Del77**, Th. finitude 3.7]. For a $\mathcal{D}$-module version, let us refer to [**Del07**]. Thus, $\operatorname{Max}\mathrm{Sl}^{\mathrm{nb}}_f(\overline{\mathbb{Q}}_\ell)$ makes sense if $\mathrm{Sl}^{\mathrm{nb}}_f(\overline{\mathbb{Q}}_\ell)$ is not empty. Otherwise, we set $\operatorname{Max}\mathrm{Sl}^{\mathrm{nb}}_f(\overline{\mathbb{Q}}_\ell) = +\infty$. Proposition 2.3.4 suggests and gives a positive answer to the following question for smooth curves

Question 2. — *Let V/k be a scheme of finite type and $\mathcal{F} \in D^b_c(V, \overline{\mathbb{Q}}_\ell)$. Is it true that the following set*

$$\{r/(1 + \operatorname{Max}\mathrm{Sl}^{\mathrm{nb}}_f(\overline{\mathbb{Q}}_\ell)),\ \text{for } r \in \mathrm{Sl}^{\mathrm{nb}}_f(\mathcal{F}) \text{ and } f \in \mathcal{O}_V\} \tag{0.0.1}$$

is bounded?

Let us explain what $\mathrm{Sl}^{\mathrm{nb}}_f(\mathcal{F})$ means in this global setting. A function $f \in \Gamma(U, \mathcal{O}_V)$ reads as $f : U \longrightarrow \mathbb{A}^1_k$. If S is the strict henselianization of $\mathbb{A}^1_k$ at a geometric point over the origin, we set $\mathrm{Sl}^{\mathrm{nb}}_f(\mathcal{F}) := \mathrm{Sl}^{\mathrm{nb}}_{f_S}(\mathcal{F}_{U_S})$ where the subscripts are synonyms of pull-back.

For smooth curves, the main point of the proof of boundedness is the concavity of Herbrand φ functions. In case f has generalized semi-stable reduction (see 1.4), the above weighted slopes are the usual nearby slopes. This is the following

1. see 2.1.2 (3) for a precise statement.

Theorem 2. — *Suppose that $f : X \longrightarrow S$ has generalized semi-stable reduction. Then we have $\mathrm{Sl}^{\mathrm{nb}}_f(\overline{\mathbb{Q}}_\ell) = \{0\}$.*

We owe the proof of this theorem to Joseph Ayoub. For the vanishing of $\mathcal{H}^0\psi^t_f$, we also give an earlier argument based on the geometric connectivity of the connected components of the moderate Milnor fibers in case of generalized semi-stable reduction.

As a possible application of a boundedness theorem in the arithmetic setting, let us remark that for every compactification $j : V \longrightarrow \overline{V}$, one could define a separated decreasing $\mathbb{R}_{\geqslant 0}$-filtration on $\pi_1(V)$ by looking for each $r \in \mathbb{R}_{\geqslant 0}$ at the category of ℓ-adic local systems L on V such that the weighted slopes (0.0.1) of $j_!L$ are $\leqslant r$.

Let us also remark that on a smooth curve C, the tameness of $\mathcal{F} \in \mathrm{Sh}_c(C, \overline{\mathbb{Q}}_\ell)$ at $0 \in C$ is characterized by $\mathrm{Sl}^{\mathrm{nb}}_f(\mathcal{F}) \subset [0, \mathrm{Max}\,\mathrm{Sl}^{\mathrm{nb}}_f(\overline{\mathbb{Q}}_\ell)]$ for every $f \in \mathcal{O}_C$ vanishing only at 0. This suggests a notion of tame complex in any dimension that may be of interest.

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1. Notations

1.1. — For a general reference on wild ramification in dimension 1, let us mention [**Ser68**]. Let η_t be the point of S corresponding to the tamely ramified closure K_t of K in $\overline{K}$ and $P_K := \mathrm{Gal}(\overline{K}/K_t)$ the wild ramification group of K. We denote by $(G^r_K)_{r \in \mathbb{R}_{\geqslant 0}}$ the *upper-numbering ramification filtration on* G_K and define

$$G^{r+}_K := \overline{\bigcup_{r'>r} G^{r'}_K}$$

If L/K is a finite extension, we denote by S_L the normalization of S in L and v_L the valuation on L associated with the maximal ideal of S_L.

If moreover L/K is separable, we denote by $q : G_K \longrightarrow G_K/G_L$ the quotient morphism and define a decreasing separated $\mathbb{R}_{\geqslant 0}$-filtration on the set G_K/G_L by $(G_K/G_L)^r := q(G^r_K)$. We also define $(G_K/G_L)^{r+} := q(G^{r+}_K)$.

In case L/K is Galois, this filtration is the upper numbering ramification filtration on $\mathrm{Gal}(L/K)$. If L/K is non separable trivial, the *jumps* of L/K are the $r \in \mathbb{R}_{\geqslant 0}$ such that $(G_K/G_L)^{r+} \subsetneq (G_K/G_L)^r$. If L/K is trivial, we say by convention that 0 is the only jump of $\mathrm{Gal}(L/K)$.

1.2. — For $M \in D^b_c(\eta, \overline{\mathbb{Q}}_\ell)$, we denote by $\mathrm{Sl}(M) \subset \mathbb{R}_{\geqslant 0}$ the set of *slopes* of M as defined in [**Kat88**, Ch 1]. We view M in an equivalent way as a continuous representation of G_K.

1.3. — Let $f : X \longrightarrow S$ be a morphism of finite type and $\mathcal{F} \in D^b_c(X_\eta, \overline{\mathbb{Q}}_\ell)$. Consider the following diagram with cartesian squares

$$\begin{tikzcd} X_s \arrow[r, "i"] \arrow[d] & X \arrow[d, "f"] & X_{\overline{\eta}} \arrow[l, "\overline{j}"'] \arrow[d] \\ s \arrow[r] & S & \overline{\eta} \arrow[l] \end{tikzcd}$$

Following [**DK73**, XIII], we define the *nearby cycles of* $\mathcal{F}$ as

$$\psi_f \mathcal{F} := i^* R\overline{j}_* \overline{j}^* \mathcal{F}$$

By [**Del77**, Th. finitude 3.2], the complex $\psi_f \mathcal{F}$ is an object of $D^b_c(X_s, \overline{\mathbb{Q}}_\ell)$ endowed with a continuous G_K-action. Define $X_t := X \times_S \eta_t$ and $j_t : X_t \longrightarrow X$ the projection. Following [**Gro72**, I.2], we define the *moderate nearby cycles of* $\mathcal{F}$ as

$$\psi^t_f \mathcal{F} := i^* Rj_{t*} j_t^* \mathcal{F}$$

It is a complex in $D^b_c(X_s, \overline{\mathbb{Q}}_\ell)$ endowed with a continuous G/P_K-action. Since P_K is a pro-p group, we have a canonical identification

$$\psi^t_f \mathcal{F} \simeq (\psi_f \mathcal{F})^{P_K}$$

Note that by proper base change [**AGV73**, XII], ψ^t_f and ψ_f are compatible with proper push-forward.

1.4. — By a *generalized semi-stable reduction* morphism, we mean a morphism $f : X \longrightarrow S$ of finite type such that etale locally on X, f has the form

$$S[x_1, \dots, x_n]/(\pi - x_1^{a_1} \cdots x_m^{a_m}) \longrightarrow S$$

where π is a uniformizer of S and where the $a_i \in \mathbb{N}^*$ are prime to p.

1.5. — If X is a scheme, $x \in X$ and if $\overline{x}$ is a geometric point of X lying over X, we denote by X^{sh}_x the strict henselization of X at x.

2. Nearby slopes in dimension one

2.1. — We show here that nearby slopes associated with the identity morphism are the usual slopes as in [**Kat88**, Ch 1].

Lemma 2.1.1. — *For every* $M \in \mathrm{Sh}_c(\eta, \overline{\mathbb{Q}}_\ell)$, *we have*

$$\mathrm{Sl}^{\mathrm{nb}}_{\mathrm{id}}(M) = \mathrm{Sl}(M)$$

Proof. — We first remark that ψ^t_{id} is just the "invariant under P" functor. Suppose that $r \in \mathrm{Sl}(M)$. Then M has a non zero quotient N purely of slope r. The dual $N^\vee$ has pure slope r. Since N is non zero, the canonical map

$$N \otimes N^\vee \longrightarrow \overline{\mathbb{Q}}_\ell$$

is surjective. Since taking P-invariants is exact, we obtain that the maps in

$$(M \otimes N^\vee)^P \longrightarrow (N \otimes N^\vee)^P \longrightarrow \overline{\mathbb{Q}}_\ell$$

are surjective. Hence $(M \otimes N^\vee)^P \neq 0$, so $r \in \mathrm{Sl}^{\mathrm{nb}}_{\mathrm{id}}(M)$.

If r is not a slope of M, then for any N of slope r, the slopes of $M \otimes N$ are non zero. This is equivalent to $(M \otimes N)^P = 0$. □

We deduce the following

Lemma 2.1.2. — *Let $f : X \longrightarrow S$ be a finite morphism with X local and $\mathcal{F} \in \mathrm{Sh}_c(X_\eta, \overline{\mathbb{Q}}_\ell)$.*

(1) $\mathrm{Sl}^{\mathrm{nb}}_f(\mathcal{F}) = \mathrm{Sl}(f_*\mathcal{F})$.

(2) Suppose that X is regular connected and let L/K be the extension of function fields induced by f. Suppose that L/K is separable. Then $\mathrm{Max}\,\mathrm{Sl}^{\mathrm{nb}}_f(\overline{\mathbb{Q}}_\ell)$ is the highest jump in the ramification filtration on G_K/G_L.

(3) Suppose further in (2) that L/K is Galois and set $G := \mathrm{Gal}(L/K)$. Then $\mathrm{Sl}^{\mathrm{nb}}_f(\overline{\mathbb{Q}}_\ell)$ is the union of $\{0\}$ with the set of jumps in the ramification filtration on G.

Proof. — Point (1) comes from 2.1.1 and the compatibility of ψ^t_f with proper push-forward.

From point (1) and $f_*\overline{\mathbb{Q}}_\ell \simeq \overline{\mathbb{Q}}_\ell[G_K/G_L]$, we deduce

$$\mathrm{Sl}^{\mathrm{nb}}_f(\overline{\mathbb{Q}}_\ell) = \mathrm{Sl}(\overline{\mathbb{Q}}_\ell[G_K/G_L])$$

If L/K is trivial, (2) is true by our definition of jumps in that case. If L/K is non trivial, $r_{\max} = \mathrm{Max}\,\mathrm{Sl}(\overline{\mathbb{Q}}_\ell[G_K/G_L])$ is characterized by the property that $G_K^{r_{\max}}$ acts non trivially on $\overline{\mathbb{Q}}_\ell[G_K/G_L]$ and $G_K^{r_{\max}+}$ acts trivially. On the other hand, the highest jump r_0 in the ramification filtration on G_K/G_L is such that $q(G_K^{r_0}) \neq \{G_L\}$ and $q(G_K^{r_0+}) = \{G_L\}$, that is $G_K^{r_0} \not\subset G_L$ and $G_K^{r_0+} \subset G_L$. The condition $G_K^{r_0} \not\subset G_L$ ensures that $G_K^{r_0}$ acts non trivially on $\overline{\mathbb{Q}}_\ell[G_K/G_L]$. If $h \in G_K^{r_0+}$, then for every $g \in G_K$

$$h \cdot (gG_L) = hgG_L = gg^{-1}hgG_L = gG_L$$

where the last equality comes from the fact that since $G_K^{r_0+}$ is a normal subgroup in G_K, we have $g^{-1}hg \in G_K^{r_0+} \subset G_L$. So (2) is proved.

Let S be the union of $\{0\}$ with the set of jumps in the ramification filtration of G. To prove (3), we have to prove $\mathrm{Sl}(\overline{\mathbb{Q}}_\ell[G]) = S$. If $r \in \mathbb{R}_{\geqslant 0}$ does not belong to S, we can find an open interval J containing r such that $G^{r'} = G^r$ for every $r' \in J$. In particular, the image of $G_K^{r'}$ by $G_K \longrightarrow \mathrm{GL}(\overline{\mathbb{Q}}_\ell[G])$ does not depend on r' for every $r' \in J$. So r is not a slope of $\overline{\mathbb{Q}}_\ell[G]$.

Reciprocally, $\overline{\mathbb{Q}}_\ell[G]$ contains a copy of the trivial representation, so $0 \in \mathrm{Sl}(\overline{\mathbb{Q}}_\ell[G])$. Let $r \in S\backslash\{0\}$. The projection morphism $G \longrightarrow G/G^{r+}$ induces a surjection of G_K-representations

$$\overline{\mathbb{Q}}_\ell[G] \longrightarrow \overline{\mathbb{Q}}_\ell[G/G^{r+}] \longrightarrow 0$$

So $\mathrm{Sl}(\overline{\mathbb{Q}}_\ell[G/G^{r+}]) \subset \mathrm{Sl}(\overline{\mathbb{Q}}_\ell[G])$. Note that G^{r+} acts trivially on $\overline{\mathbb{Q}}_\ell[G/G^{r+}]$. By definition $G^{r+} \subsetneq G^r$, so G^r acts non trivially on $\overline{\mathbb{Q}}_\ell[G/G^{r+}]$. So $r = \mathrm{Max}\,\mathrm{Sl}(\overline{\mathbb{Q}}_\ell[G/G^{r+}])$ and point (3) is proved.

$\square$

2.2. — Let us draw a consequence of 2.1.1. We suppose that $f : X \longrightarrow S$ is proper. Let $\mathcal{F} \in D^b_c(X_\eta, \overline{\mathbb{Q}}_\ell)$. The G_K-module associated to $R^k f_* \mathcal{F} \in D^b_c(\eta, \overline{\mathbb{Q}}_\ell)$ is $H^k(X_{\overline{\eta}}, \mathcal{F})$. From 2.1.1, we deduce

$$\begin{aligned} \mathrm{Sl}(H^k(X_{\overline{\eta}}, \mathcal{F})) &= \mathrm{Sl}^{\mathrm{nb}}_{\mathrm{id}}(R^k f_* \mathcal{F}) \\ &\subset \mathrm{Sl}^{\mathrm{nb}}_{\mathrm{id}}(R f_* \mathcal{F}) \end{aligned}$$

where the inclusion comes from the fact that taking P_K-invariants is exact. For every $N \in \mathrm{Sh}_c(\eta, \overline{\mathbb{Q}}_\ell)$, the projection formula and the compatibility of ψ^t_f with proper push-forward gives

$$\begin{aligned} \psi^t_{\mathrm{id}}(R f_* \mathcal{F} \otimes N) &\simeq \psi^t_{\mathrm{id}}(R f_*(\mathcal{F} \otimes f^* N)) \\ &\simeq R f_* \psi^t_f(\mathcal{F} \otimes f^* N) \end{aligned}$$

Hence we have proved the following

Proposition 2.2.1. — *Let $f : X \longrightarrow S$ be a proper morphism, and let $\mathcal{F} \in D^b_c(X_\eta, \overline{\mathbb{Q}}_\ell)$. For every $i \geqslant 0$, we have*

$$\mathrm{Sl}(H^i(X_{\overline{\eta}}, \mathcal{F})) \subset \mathrm{Sl}^{\mathrm{nb}}_f(\mathcal{F})$$

2.3. Boundedness. — We first need to see that the upper-numbering filtration is unchanged by purely inseparable base change. This is the following

Lemma 2.3.1. — *Let K'/K be a purely inseparable extension of degree p^n. Let L/K be finite Galois extension, $L' := K' \otimes_K L$ the associated Galois extension of K'. Then, the isomorphism*

$$\mathrm{Gal}(L/K) \xrightarrow{\sim} \mathrm{Gal}(L'/K') \tag{2.3.2}$$

$$g \longrightarrow \mathrm{id} \otimes g \tag{2.3.3}$$

is compatible with the upper-numbering filtration.

Proof. — Note that for every $g \in \mathrm{Gal}(L/K)$, $\mathrm{id} \otimes g \in \mathrm{Gal}(L'/K')$ is determined by the property that its restriction to L is g.

Let π be a uniformizer of S and π_L a uniformizer of S_L. We have $K \simeq k((\pi))$ and $L \simeq k((\pi_L))$. Since k is perfect and since K'/K and L'/L are purely inseparable of degree p^n, we have $K' = k((\pi^{1/p^n}))$ and $L' = k((\pi_L^{1/p^n}))$. So π_L^{1/p^n} is a uniformizer of $S_{L'}$. For every $\sigma \in \mathrm{Gal}(L'/K')$ we have

$$(\sigma(\pi_L^{1/p^n}) - \pi_L^{1/p^n})^{p^n} = \sigma_{|L}(\pi_L) - \pi_L$$

so

$$\begin{aligned} v_{L'}(\sigma(\pi_L^{1/p^n}) - \pi_L^{1/p^n}) &= \frac{1}{p^n} v_{L'}(\sigma_{|L}(\pi_L) - \pi_L) \\ &= v_L(\sigma_{|L}(\pi_L) - \pi_L) \end{aligned}$$

So (2.3.2) commutes with the lower-numbering filtration. Hence, (2.3.2) commutes with the upper-numbering filtration and lemma 2.3.1 is proved. □

Boundedness in case of smooth curves over k is a consequence of the following

Proposition 2.3.4. — *Let S_0 be an henselian trait over k, let $\eta_0 = \operatorname{Spec} K_0$ be the generic point of S_0 and $M \in \operatorname{Sh}_c(\eta_0, \overline{\mathbb{Q}}_\ell)$. There exists a constant $C_M \geqslant 0$ depending only on M such that for every finite morphism $f : S_0 \longrightarrow S$, we have*

$$\operatorname{Sl}_f^{\mathrm{nb}}(M) \subset [0, \operatorname{Max}(C_M, \operatorname{Max} \operatorname{Sl}_f^{\mathrm{nb}}(\overline{\mathbb{Q}}_\ell))] \tag{2.3.5}$$

In particular, the quantity

$$\operatorname{Max} \operatorname{Sl}_f^{\mathrm{nb}}(M)/(1 + \operatorname{Max} \operatorname{Sl}_f^{\mathrm{nb}}(\overline{\mathbb{Q}}_\ell))$$

is bounded uniformely in f.

Proof. — By 2.1.2 (1), we have to bound $\operatorname{Sl}(f_* M)$ in terms of $\operatorname{Max} \operatorname{Sl}(f_* \overline{\mathbb{Q}}_\ell)$. Using [**Kat88**, I 1.10], we can replace $\overline{\mathbb{Q}}_\ell$ by $\mathbb{F}_\lambda$, where $\lambda = \ell^n$. Hence, G_{K_0} acts on M via a finite quotient $H \subset \mathrm{GL}_{\mathbb{F}_\lambda}(M)$. Let L/K_0 be the corresponding finite Galois extension and $f_M : S_L \longrightarrow S_0$ the induced morphism. We have $H = \operatorname{Gal}(L/K_0)$. Let us denote by r_M the highest jump in the ramification filtration of H. Using Herbrand functions [**Ser68**, IV 3], we will prove that the constant $C_M := \psi_{L/K_0}(r_M)$ does the job.

Using 2.3.1, we are left to treat the case where K_0/K is separable. The adjunction morphism

$$M \longrightarrow f_{M*} f_M^* M$$

is injective. Since $f_M^* M \simeq \mathbb{F}_\lambda^{\operatorname{rg} M}$, we obtain by applying f_* an injection

$$f_* M \longrightarrow \mathbb{F}_\lambda[\operatorname{Gal}(L/K)]^{\operatorname{rg} M}$$

So we are left to bound the slopes of $\mathbb{F}_\lambda[\operatorname{Gal}(L/K)]$ viewed as a G_K-representation, that is by 2.1.2 (2) the highest jump in the upper-numbering ramification filtration of $\operatorname{Gal}(L/K)$. By 2.1.2 (2), $r_0 := \operatorname{Max} \operatorname{Sl}_f^{\mathrm{nb}}(\overline{\mathbb{Q}}_\ell)$ is the highest jump in the ramification filtration of $\operatorname{Gal}(L/K)/H$. Choose $r > \operatorname{Max}(r_0, \varphi_{L/K}\psi_{L/K_0}(r_M))$. We have

$$\begin{aligned} \operatorname{Gal}(L/K)^r &= H \cap \operatorname{Gal}(L/K)^r \\ &= H \cap \operatorname{Gal}(L/K)_{\psi_{L/K}(r)} \\ &= H_{\psi_{L/K}(r)} \\ &= H^{\varphi_{L/K_0}\psi_{L/K}(r)} \\ &= \{1\} \end{aligned}$$

The first equality comes from $r > r_0$. The third equality comes from the compatibility of the lower-numbering ramification filtration with subgroups. The last equality comes from the fact that $r > \varphi_{L/K}\psi_{L/K_0}(r_M)$ is equivalent to $\varphi_{L/K_0}\psi_{L/K}(r) > r_M$. Hence,

$$\mathrm{Sl}_f^{\mathrm{nb}}(M) \subset [0, \mathrm{Max}(r_0, \varphi_{L/K}\psi_{L/K_0}(r_M))]$$

Since $\varphi_{L/K} : [-1, +\infty[\longrightarrow \mathbb{R}$ is concave, satisfies $\varphi_{L/K}(0) = 0$ and is equal to the identity on $[-1, 0]$, we have

$$\varphi_{L/K}\psi_{L/K_0}(r_M) \leqslant \psi_{L/K_0}(r_M)$$

and we obtain (2.3.5) by setting $C_M := \psi_{L/K_0}(r_M)$. □

3. Proof of Theorem 1

3.1. Preliminary. — Let us consider the affine line $\mathbb{A}^1_S \longrightarrow S$ over S. Let s' be the generic point of $\mathbb{A}^1_s$ and S' the strict henselianization of $\mathbb{A}^1_S$ at s'. We denote by $\overline{S}$ the normalization of S in $\overline{\eta}$, by κ the function field of the strict henselianization of $\mathbb{A}^1_{\overline{S}}$ at s', and by $\overline{\kappa}$ an algebraic closure of κ. We have $\kappa \simeq K' \otimes_K \overline{K}$ and

$$G_K \simeq \mathrm{Gal}(\kappa/K') \tag{3.1.1}$$

Let L/K be a finite Galois extension of K in $\overline{K}$. Set $L' := K' \otimes_K L$. At finite level, (3.1.1) reads

$$\mathrm{Gal}(L/K) \xrightarrow{\sim} \mathrm{Gal}(L'/K') \tag{3.1.2}$$

$$g \longrightarrow \mathrm{id} \otimes g \tag{3.1.3}$$

Since a uniformizer in S_L is also a uniformizer in $S'_{L'}$, we deduce that (3.1.2) is compatible with the lower-numbering ramification filtration on $\mathrm{Gal}(L/K)$ and $\mathrm{Gal}(L'/K')$. Hence, (3.1.2) is compatible with the upper-numbering ramification filtration on $\mathrm{Gal}(L/K)$ and $\mathrm{Gal}(L'/K')$. We deduce that through (3.1.1), the canonical surjection $G_{K'} \longrightarrow G_K$ is compatible with the upper-numbering ramification filtration.

3.2. The proof. — We can suppose that $\mathcal{F}$ is concentrated in degree 0. In case $\dim X = 0$, there is nothing to prove. We first reduce the proof of Theorem 1 to the case where $\dim X = 1$ by arguing by induction on $\dim X$.

Since the problem is local on X, we can suppose that X is affine. We thus have a digram

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & \mathbb{A}^n_S \arrow[r, hook] \arrow[d] & \mathbb{P}^n_S \arrow[dl] \\ & S & \end{tikzcd} \tag{3.2.1}$$

Let $\overline{X}$ be the closure of X in $\mathbb{P}^n_S$ and let $j : X \hookrightarrow \overline{X}$ be the associated open immersion. Replacing $(X, \mathcal{F})$ by $(\overline{X}, j_!\mathcal{F})$, we can suppose X/S projective. Then Theorem 1 is a consequence of the following assertions

(A) There exists a finite set $E_A \subset \mathbb{R}_{\geqslant 0}$ such that for every $N \in \mathrm{Sh}_c(\eta, \overline{\mathbb{Q}}_\ell)$ with slope not in E_A, the support of $\psi_f^t(\mathcal{F} \otimes f^*N)$ is punctual.

(B) There exists a finite set $E_B \subset \mathbb{R}_{\geqslant 0}$ such that for every $N \in \mathrm{Sh}_c(\eta, \overline{\mathbb{Q}}_\ell)$ with slope not in E_B, we have

$$R\Gamma(X_s, \psi_f^t(\mathcal{F} \otimes f^*N)) \simeq 0$$

Let us prove (A). This is a local statement on X, so we can suppose X to be a closed subset in $\mathbb{A}^n_S$ and consider the factorisations

$$\begin{tikzcd} X \arrow[r, "p_i"] \arrow[dr, "f"'] & \mathbb{A}^1_S \arrow[d] \\ & S \end{tikzcd}$$

where p_i is the projection on the i-th factor of $\mathbb{A}^n_S$. Using the notations in 3.1, let X'/S' making the upper square of the following diagram

$$\begin{tikzcd} X' \arrow[r, "\lambda"] \arrow[d, "p_i'"'] & X \arrow[d, "p_i"] \\ S' \arrow[r] \arrow[dr, "h"'] & \mathbb{A}^1_S \arrow[d] \\ & S \end{tikzcd}$$

cartesian. Let us set $\mathcal{F}' := \lambda^*\mathcal{F}$ and $N' := h^*N$. From [**Del77**, Th. finitude 3.4], we have

$$\lambda^*\psi_f(\mathcal{F} \otimes f^*N) \simeq \psi_{hp_i'}(\mathcal{F}' \otimes p_i'^*N') \simeq \psi_{p_i'}(\mathcal{F}' \otimes p_i'^*N')^{G_\kappa} \tag{3.2.2}$$

where G_κ is a pro-p group sitting in an exact sequence

$$1 \longrightarrow G_\kappa \longrightarrow G_{K'} \longrightarrow G_K \longrightarrow 1$$

In particular, G_κ is a subgroup of the wild-ramification group $P_{K'}$ of $G_{K'}$. So applying the $P_{K'}$-invariants on (3.2.2) yields

$$\lambda^*\psi_f^t(\mathcal{F} \otimes f^*N) \simeq \psi_{p_i'}^t(\mathcal{F}' \otimes p_i'^*N') \tag{3.2.3}$$

If N has pure slope r, we know from 3.1 that N' has pure slope r as a sheaf on η'. Applying the recursion hypothesis gives a finite set $E_i \subset \mathbb{R}_{\geqslant 0}$ such that the right-hand side of (3.2.3) is 0 for N of slope not in E_i. The union of the E_i for $1 \leqslant i \leqslant n$ is the set E_A sought for in (A).

To prove (B), we observe that the compatibility of ψ_f^t with proper morphisms and the projection formula give

$$R\Gamma(X_s, \psi_f^t(\mathcal{F} \otimes f^*N)) \simeq \psi_{\mathrm{id}}^t(Rf_*\mathcal{F} \otimes N)$$

By 2.1.1, the set $E_B := \mathrm{Sl}(Rf_*\mathcal{F})$ has the required properties.

We are thus left to prove Theorem 1 in the case where $\dim X = 1$. At the cost of localizing, we can suppose that X is local and maps surjectively on S. Let x be the closed point of X. Note that $k(x)/k(s)$ is of finite type but may not be finite. Choosing a transcendence basis of $k(x)/k(s)$ yields a factorization $X \longrightarrow S' \longrightarrow S$

satisfying $\mathrm{trdeg}_{k(s')}\, k(x) = \mathrm{trdeg}_{k(s)}\, k(x) - 1$.

So we can further suppose that $k(x)/k(s)$ is finite. Since $k(s)$ is algebraically closed, we have $k(x) = k(s)$. If $\widehat{S}$ denotes the completion of S at s, we deduce that $X \times_S \widehat{S}$ is finite over $\widehat{S}$. By faithfully flat descent [**Gro71**, VIII 5.7], we obtain that X/S is finite. We conclude the proof of Theorem 1 with 2.1.2 (1).

4. Proof of Theorem 2

4.1. — That $0 \in \mathrm{Sl}_f(\overline{\mathbb{Q}}_\ell)$ is easy by looking at the smooth locus of f. We are left to prove that for every $N \in \mathrm{Sh}_c(\eta, \overline{\mathbb{Q}}_\ell)$ with slope > 0, the following holds

$$\psi_f^t f^* N \simeq 0 \tag{4.1.1}$$

Since the problem is local on X for the étale topology, we can suppose that $X = S[x_1, \dots, x_n]/(\pi - x_1^{a_1} \cdots x_m^{a_m})$ and we have to prove (4.1.1) at the origin $0 \in X_s$. Let a be the lowest commun multiple of the a_i and define $b_i = a/a_i$. Note that a and the b_i are prime to p. Hence the morphism h defined as

$$\begin{array}{rcl} Y := S[t_1, \dots, t_n]/(\pi - t_1^a \cdots t_m^a) & \longrightarrow & X \\ (t_1, \dots, t_n) & \longrightarrow & (t_1^{b_1}, \dots, t_m^{b_m}, t_{m+1}, \dots, t_n) \end{array}$$

is finite surjective and finite etale above η with Galois group G. Set $g = fh$. Then

$$(\mathcal{H}^i \psi_f^t f^* N)_{\overline{0}} \simeq (\mathcal{H}^i \psi_g^t g^* N)_{\overline{0}}^G$$

for every $i \geqslant 0$, so we can suppose $a_1 = \cdots = a_m = a$. Since a is prime to p, the map of absolute Galois groups induced by $S[\pi^{1/a}] \longrightarrow S$ induces an identification at the level of the ramification groups. By compatibility of nearby cycles with change of trait [**Del77**, Th. finitude 3.7], we can suppose $a = 1$.

Let us now reduce the proof of Theorem 2 to the case where $m = 1$. We argue by induction on m. The case $m = 1$ follows from the compatibility of nearby cycles with smooth morphisms. We thus suppose that Theorem 2 is true for $m < n$ with all a_i equal to 1 and prove it for $m + 1$ with all a_i equal to 1. Let $h : \widetilde{X} \longrightarrow X$ be the blow-up of X along $x_m = x_{m+1} = 0$. Define $g := fh$ and denote by E the exceptional divisor of $\widetilde{X}$. Since h induces an isomorphism on the generic fibers, and since ψ_f^t is compatible with proper push-forward, we have

$$Rh_* \psi_g^t g^* N \simeq \psi_f^t f^* N \simeq 0 \tag{4.1.2}$$

By proper base change, (4.1.2) gives

$$R\Gamma(h^{-1}(0), (\psi_g^t g^* N)_{|h^{-1}(0)}) \simeq 0 \tag{4.1.3}$$

The scheme $\widetilde{X}$ is covered by a chart U affine over S given by

$$S[(u_i)_{1 \leqslant i \leqslant n}]/(\pi - u_1 \cdots u_m)$$

with $E \cap U$ given by $u_m = 0$, and a chart U' affine over S given by

$$S[(u_i)_{1 \leqslant i \leqslant n}]/(\pi - u_1 \cdots u_{m+1})$$

with $E \cap U'$ given by $u_{m+1} = 0$. By recursion hypothesis, $(\psi_g^t g^* N)_{|h^{-1}(0)}$ is a sky-scraper sheaf supported at the origin 0 of U'. Hence, (4.1.3) gives

$$(\psi_g^t g^* N)_{\overline{0}} \simeq 0$$

This finishes the induction, and thus the proof of Theorem 2.

4.2. — Let us give a geometric-flavoured proof of

$$\mathcal{H}^0 \psi_f^t f^* N \simeq 0$$

in case $X = S[x_1, \dots, x_n]/(\pi - x_1^{a_1} \cdots x_m^{a_m})$. By constructibility [**Del77**, Th. finitude 3.2], it is enough to work at the level of germs at a geometric point $\overline{x}$ lying over a closed point $x \in X$.

Hence, we have to prove $H^0(C, f^*N) \simeq 0$ for every connected component C of $X^{\mathrm{sh}}_{x,\eta_t}$. For such C, denote by $\rho_C : \pi_1(C) \longrightarrow \pi_1(\eta_t) = P_K$ the induced map. Then $H^0(C, f^*N) \simeq N^{\mathrm{Im}\,\rho_C}$. Since by definition $N^{P_K} = 0$, it is enough to prove that ρ_C is surjective. From V 6.9 and IX 3.4 of [**Gro71**], we are left to prove that C is geometrically connected. To do this, we can always replace X^{sh}_x by its formalization $\widehat{X}_x = \operatorname{Spec} R[\![\underline{x}]\!]/(\pi - x_1^{a_1} \cdots x_m^{a_m})$.

By hypothesis, $d := \gcd(a_1, \dots, a_m)$ is prime to p, so π has a d-root in K_t. Hence $\widehat{X}_{x,\eta_t}$ is a direct union of d copies of

$$\operatorname{Spec} K_t \otimes_R R[\![\underline{x}]\!]/(\pi^{1/d} - x_1^{a'_1} \cdots x_m^{a'_m})$$

where $a_i = da'_i$. So we have to prove the following

Lemma 4.2.1. — *Let* $a_1, \dots, a_m, d \in \mathbb{N}^*$ *with* $\gcd(a_1, \dots, a_m) = 1$. *Then*

$$\operatorname{Spec} \overline{K} \otimes_R R[\![\underline{x}]\!]/(\pi^{1/d} - x_1^{a_1} \cdots x_m^{a_m}) \tag{4.2.2}$$

is connected.

Proof. — One easily reduces to the case $d = 1$. If R' is the normalization of R in a Galois extension of K in $\overline{K}$, it is enough to prove that $\operatorname{Spec} R'[\![\underline{x}]\!]/(\pi - x_1^{a_1} \cdots x_m^{a_m})$ is irreducible. If π' is a uniformizer of R', we have $R' \simeq k[\![\pi']\!]$, we write $\pi = P(\pi')$ where $P \in k[\![X]\!]$ and then we are left to prove that $f_{a,P} := P(\pi') - x_1^{a_1} \cdots x_m^{a_m}$ is irreducible in $k[\![x_1, \dots, x_n, \pi']\!]$. This follows from $\gcd(a_1, \dots, a_m) = 1$ via Lypkovski's indecomposability criterion [**Lip88**, 2.10] for the Newton polyhedron associated to $f_{a,P}$.

□

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